Exercise 4

Verify that the entries for $\nabla^2 \mathbf{v}$ in Table A.7-2 can be obtained by any one of the following methods:

(a) First verify that, in cylindrical coordinates the operator $(\nabla \cdot \nabla)$ is

$$(\nabla \cdot \nabla) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{A.7-36}$$

and then apply the operator to $\mathbf{v}$.

(b) Use the expression for $[\nabla \cdot \boldsymbol{\tau}]$ in Table A.7-2, but substitute the components for $\nabla \mathbf{v}$ in place of the components of $\boldsymbol{\tau}$, so as to obtain $[\nabla \cdot \nabla \mathbf{v}]$.

(c) Use Eq. A.4-22:

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - [\nabla \times [\nabla \times \mathbf{v}]] \tag{A.7-37}$$

and use the gradient, divergence, and curl operations in Table A.7-2 to evaluate the operations on the right side.

Solution

In cylindrical coordinates the nabla operator is

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}.$$

The partial derivatives of $\boldsymbol{\delta}_r$, $\boldsymbol{\delta}_\theta$, and $\boldsymbol{\delta}_z$ in cylindrical coordinates are given by equations A.7-1, A.7-2, and A.7-3,

$$\frac{\partial \boldsymbol{\delta}_r}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial r} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial r} = 0 \tag{A.7-1}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial \theta} = \boldsymbol{\delta}_\theta \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta} = -\boldsymbol{\delta}_r \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial \theta} = 0 \tag{A.7-2}$$

$$\frac{\partial \boldsymbol{\delta}_r}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_\theta}{\partial z} = 0 \qquad \frac{\partial \boldsymbol{\delta}_z}{\partial z} = 0. \tag{A.7-3}$$

Part (a)

$$\begin{aligned}
\nabla \cdot \nabla &= \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \\
&= \boldsymbol{\delta}_r \cdot \frac{\partial}{\partial r}\left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right) \\
&\quad + \boldsymbol{\delta}_z \cdot \frac{\partial}{\partial z}\left(\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right)
\end{aligned}$$

Apply the product rule.

$$\begin{aligned}
&= \boldsymbol{\delta}_r \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial r}\frac{\partial}{\partial r} + \boldsymbol{\delta}_r \frac{\partial^2}{\partial r^2} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} - \boldsymbol{\delta}_\theta \frac{1}{r^2}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} + \frac{\partial \boldsymbol{\delta}_z}{\partial r}\frac{\partial}{\partial z} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial r \partial z} \right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial \theta}\frac{\partial}{\partial r} + \boldsymbol{\delta}_r \frac{\partial^2}{\partial \theta \partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + \frac{\partial \boldsymbol{\delta}_z}{\partial \theta}\frac{\partial}{\partial z} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial \theta \partial z} \right) \\
&\quad + \boldsymbol{\delta}_z \cdot \left(\frac{\partial \boldsymbol{\delta}_r}{\partial z}\frac{\partial}{\partial r} + \boldsymbol{\delta}_r \frac{\partial^2}{\partial z \partial r} + \frac{\partial \boldsymbol{\delta}_\theta}{\partial z}\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial z \partial \theta} + \frac{\partial \boldsymbol{\delta}_z}{\partial z}\frac{\partial}{\partial z} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial z^2} \right)
\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3 here.

$$\begin{aligned}
&= \boldsymbol{\delta}_r \cdot \left(\boldsymbol{\delta}_r \frac{\partial^2}{\partial r^2} - \boldsymbol{\delta}_\theta \frac{1}{r^2}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial r \partial z} \right) \\
&\quad + \frac{\boldsymbol{\delta}_\theta}{r} \cdot \left(\boldsymbol{\delta}_\theta \frac{\partial}{\partial r} + \boldsymbol{\delta}_r \frac{\partial^2}{\partial \theta \partial r} - \boldsymbol{\delta}_r \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial \theta^2} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial \theta \partial z} \right) \\
&\quad + \boldsymbol{\delta}_z \cdot \left(\boldsymbol{\delta}_r \frac{\partial^2}{\partial z \partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial^2}{\partial z \partial \theta} + \boldsymbol{\delta}_z \frac{\partial^2}{\partial z^2} \right)
\end{aligned}$$

Evaluate the dot products.

$$\begin{aligned}
&= \frac{\partial^2}{\partial r^2} \\
&\quad + \frac{1}{r} \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2} \right) \\
&\quad + \frac{\partial^2}{\partial z^2}
\end{aligned}$$

Therefore, in cylindrical coordinates

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

An arbitrary vector in this coordinate system can be represented as

$$\mathbf{v} = \boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z.$$

If $(\nabla \cdot \nabla) = \nabla^2$ acts on this vector, we get

$$\begin{aligned}
\nabla^2 \mathbf{v} &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) (\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&= \frac{\partial^2}{\partial r^2}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&\quad + \frac{1}{r}\frac{\partial}{\partial r}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&\quad + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z) \\
&\quad + \frac{\partial^2}{\partial z^2}(\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z)
\end{aligned}$$

Apply the product rule for one derivative.

$$\begin{aligned}\nabla^2\mathbf{v} &= \frac{\partial}{\partial r}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial r}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial\boldsymbol{\delta}_z}{\partial r}v_z + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial r}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial\boldsymbol{\delta}_z}{\partial r}v_z + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial\theta}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_z}{\partial\theta}v_z + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial\theta}\right)\\
&\quad + \frac{\partial}{\partial z}\left(\frac{\partial\boldsymbol{\delta}_r}{\partial z}v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial z} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial z}v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial z} + \frac{\partial\boldsymbol{\delta}_z}{\partial z}v_z + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial z}\right)\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3.

$$\begin{aligned}&= \frac{\partial}{\partial r}\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r}\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\boldsymbol{\delta}_\theta v_r + \boldsymbol{\delta}_r\frac{\partial v_r}{\partial\theta} - \boldsymbol{\delta}_r v_\theta + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial\theta} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial\theta}\right)\\
&\quad + \frac{\partial}{\partial z}\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial z}\right)\end{aligned}$$

Use the product rule again.

$$\begin{aligned}&= \frac{\partial\boldsymbol{\delta}_r}{\partial r}\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial r^2} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial r}\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial r^2} + \frac{\partial\boldsymbol{\delta}_z}{\partial r}\frac{\partial v_z}{\partial r} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial r^2}\\
&\quad + \frac{1}{r}\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r^2}\left(\frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}v_r + \boldsymbol{\delta}_\theta\frac{\partial v_r}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_r}{\partial\theta}\frac{\partial v_r}{\partial\theta} + \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial\theta^2} - \frac{\partial\boldsymbol{\delta}_r}{\partial\theta}v_\theta - \boldsymbol{\delta}_r\frac{\partial v_\theta}{\partial\theta} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial\theta}\frac{\partial v_\theta}{\partial\theta} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial\theta^2}\right.\\
&\qquad\qquad \left. + \frac{\partial\boldsymbol{\delta}_z}{\partial\theta}\frac{\partial v_z}{\partial\theta} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial\theta^2}\right)\\
&\quad + \frac{\partial\boldsymbol{\delta}_r}{\partial z}\frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial z^2} + \frac{\partial\boldsymbol{\delta}_\theta}{\partial z}\frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial z^2} + \frac{\partial\boldsymbol{\delta}_z}{\partial z}\frac{\partial v_z}{\partial z} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial z^2}\end{aligned}$$

Use equations A.7-1, A.7-2, and A.7-3.

$$\begin{aligned}&= \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial r^2} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial r^2} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial r^2}\\
&\quad + \frac{1}{r}\left(\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_z\frac{\partial v_z}{\partial r}\right)\\
&\quad + \frac{1}{r^2}\left(-\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta\frac{\partial v_r}{\partial\theta} + \boldsymbol{\delta}_\theta\frac{\partial v_r}{\partial\theta} + \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial\theta^2} - \boldsymbol{\delta}_\theta v_\theta - \boldsymbol{\delta}_r\frac{\partial v_\theta}{\partial\theta} - \boldsymbol{\delta}_r\frac{\partial v_\theta}{\partial\theta} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial\theta^2} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial\theta^2}\right)\\
&\quad + \boldsymbol{\delta}_r\frac{\partial^2 v_r}{\partial z^2} + \boldsymbol{\delta}_\theta\frac{\partial^2 v_\theta}{\partial z^2} + \boldsymbol{\delta}_z\frac{\partial^2 v_z}{\partial z^2}\end{aligned}$$

Factor the unit vectors.

$$\begin{aligned}\nabla^2\mathbf{v} = \boldsymbol{\delta}_r &\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) \\ &+ \boldsymbol{\delta}_\theta \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2}\right) \\ &+ \boldsymbol{\delta}_z \left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right)\end{aligned}$$

The expression

$$\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2}$$

can be written compactly as

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right],$$

and the expression

$$\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}$$

can be written compactly as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right).$$

Therefore,

$$\begin{aligned}\nabla^2\mathbf{v} = \boldsymbol{\delta}_r &\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right] + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right\} \\ &+ \boldsymbol{\delta}_\theta \left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right] + \frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2} + 2\frac{\partial v_r}{\partial \theta}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right\} \\ &+ \boldsymbol{\delta}_z \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right].\end{aligned}$$

Part (b)

Looking up the formula for $\nabla \cdot \boldsymbol{\tau}$ in Table A.7-2 on pg. 834, we have

$$\begin{aligned}\nabla \cdot \boldsymbol{\tau} = \boldsymbol{\delta}_r &\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] \\ &+ \boldsymbol{\delta}_\theta \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] \\ &+ \boldsymbol{\delta}_z \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right].\end{aligned}$$

The formula for $\nabla\mathbf{v}$ is in the same table on pg. 835.

$$\begin{aligned}\nabla\mathbf{v} = \boldsymbol{\delta}_r\boldsymbol{\delta}_r\frac{\partial v_r}{\partial r} &+ \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z\frac{\partial v_z}{\partial r} \\ &+ \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z\frac{1}{r}\frac{\partial v_z}{\partial \theta} \\ &+ \boldsymbol{\delta}_z\boldsymbol{\delta}_r\frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta\frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z\frac{\partial v_z}{\partial z}\end{aligned}$$

Substitute the components of $\nabla\mathbf{v}$ for the components of $\boldsymbol{\tau}$ in $\nabla\cdot\boldsymbol{\tau}$.

$$\begin{aligned}\nabla\cdot\nabla\mathbf{v} = \boldsymbol{\delta}_r&\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right)+\frac{\partial}{\partial z}\frac{\partial v_r}{\partial z}-\frac{1}{r}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)\right]\\ &+\boldsymbol{\delta}_\theta\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right)+\frac{\partial}{\partial z}\frac{\partial v_\theta}{\partial z}+\frac{\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r}\right)-\frac{\partial v_\theta}{\partial r}}{r}\right]\\ &+\boldsymbol{\delta}_z\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r}\frac{\partial v_z}{\partial\theta}+\frac{\partial}{\partial z}\frac{\partial v_z}{\partial z}\right]\end{aligned}$$

Expand the terms.

$$\begin{aligned}\nabla^2\mathbf{v} = \boldsymbol{\delta}_r&\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right)\\ &+\boldsymbol{\delta}_\theta\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}\right)\\ &+\boldsymbol{\delta}_z\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\end{aligned}$$

This is the same formula in part (a), so we get the same result. Therefore,

$$\begin{aligned}\nabla^2\mathbf{v} = \boldsymbol{\delta}_r&\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right]+\frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial\theta^2}-2\frac{\partial v_\theta}{\partial\theta}\right)+\frac{\partial^2 v_r}{\partial z^2}\right\}\\ &+\boldsymbol{\delta}_\theta\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right]+\frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial\theta^2}+2\frac{\partial v_r}{\partial\theta}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right\}\\ &+\boldsymbol{\delta}_z\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right].\end{aligned}$$

Part (c)

From Table A.7-2 on pg. 834, we have for the divergence of $\mathbf{v}$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z},$$

for the curl of $\mathbf{v}$

$$\begin{aligned}\nabla\times\mathbf{v} = \boldsymbol{\delta}_r&\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)\\ &+\boldsymbol{\delta}_\theta\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\\ &+\boldsymbol{\delta}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right],\end{aligned}$$

and for the gradient of s, a scalar,

$$\nabla s = \boldsymbol{\delta}_r\frac{\partial s}{\partial r}+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial s}{\partial\theta}+\boldsymbol{\delta}_z\frac{\partial s}{\partial z}.$$

Put the formulas for the gradient and divergence together.

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{v}) &= \boldsymbol{\delta}_r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}\right]\\ &\quad+\boldsymbol{\delta}_z\frac{\partial}{\partial z}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}\right]\end{aligned}$$

Expand each of the terms.

$$\begin{aligned}&=\boldsymbol{\delta}_r\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}+\frac{\partial^2 v_z}{\partial r\partial z}\right)\\ &\quad+\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)\\ &\quad+\boldsymbol{\delta}_z\left(\frac{\partial^2 v_r}{\partial z\partial r}+\frac{1}{r}\frac{\partial v_r}{\partial z}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial z\partial\theta}+\frac{\partial^2 v_z}{\partial z^2}\right)\end{aligned}$$

Now obtain the formula for $\nabla\times[\nabla\times\mathbf{v}]$ by plugging the components of $\nabla\times\mathbf{v}$ into themselves.

$$\begin{aligned}\nabla\times[\nabla\times\mathbf{v}] &= \boldsymbol{\delta}_r\left\{\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]-\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\right\}\\ &\quad+\boldsymbol{\delta}_\theta\left\{\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)-\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right]\right\}\\ &\quad+\boldsymbol{\delta}_z\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\right]-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)\right\}\end{aligned}$$

Expand each of the terms.

$$\begin{aligned}&=\boldsymbol{\delta}_r\left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{\partial^2 v_r}{\partial z^2}+\frac{\partial^2 v_z}{\partial z\partial r}\right)\\ &\quad+\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial^2 v_z}{\partial z\partial\theta}-\frac{\partial^2 v_\theta}{\partial z^2}-\frac{\partial^2 v_\theta}{\partial r^2}+\frac{v_\theta}{r^2}-\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}\right)\\ &\quad+\boldsymbol{\delta}_z\left(\frac{\partial^2 v_r}{\partial r\partial z}-\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial z}-\frac{1}{r}\frac{\partial v_z}{\partial r}-\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial z}\right)\end{aligned}$$

Subtract the two expressions to get $\nabla^2\mathbf{v}$. The second partial derivatives are equal by Clairaut's theorem, assuming they are continuous.

$$\begin{aligned}\nabla^2\mathbf{v} &= \boldsymbol{\delta}_r\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right)\\ &\quad+\boldsymbol{\delta}_\theta\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}\right)\\ &\quad+\boldsymbol{\delta}_z\left(\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\end{aligned}$$

This is the same formula in part (a), so we get the same result. Therefore,

$$\begin{aligned}\nabla^2 \mathbf{v} = \boldsymbol{\delta}_r &\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right] + \frac{1}{r^2}\left(\frac{\partial^2 v_r}{\partial \theta^2} - 2\frac{\partial v_\theta}{\partial \theta}\right) + \frac{\partial^2 v_r}{\partial z^2}\right\} \\ &+ \boldsymbol{\delta}_\theta \left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right] + \frac{1}{r^2}\left(\frac{\partial^2 v_\theta}{\partial \theta^2} + 2\frac{\partial v_r}{\partial \theta}\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right\} \\ &+ \boldsymbol{\delta}_z \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right].\end{aligned}$$